

Lecture 4 (Feb 8, 2016)

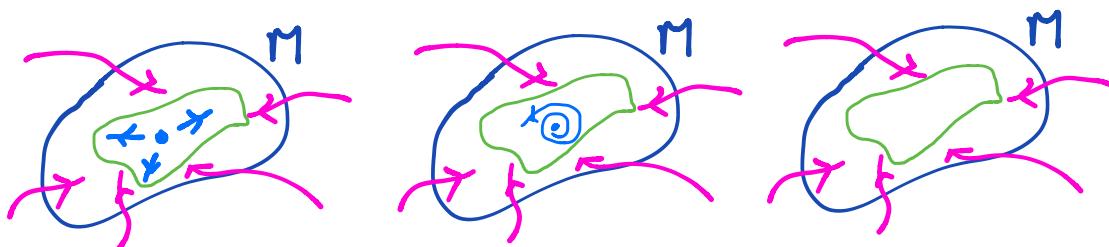
Poincaré Bendixson Criterion $\dot{x} = f(x) \neq 0$

Intuition: Bounded trajectories have to approach eq. pt or periodic orbits as $t \rightarrow \infty$.

Lemma 2.1 Let $M \subseteq \mathbb{R}^2$ be closed, bounded s.t.

- 1) M contains no eq. pts. or contains only one eq. pt. s.t. the Jacobian matrix at this point has eigenvalues with positive real parts (unstable focus or node)
- 2) Every trajectory starting in M staying in M for all future time (trapping region)

Then M contains a periodic orbit of (x) .



□ Useful tool for finding invariant region M is to compute which way vector field $f(x)$ points along level sets $V(x) = c$ for a C^1 function V .

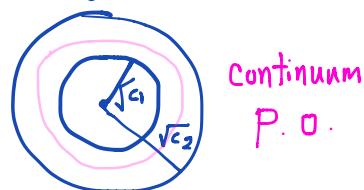
Looking for a region of the form $M = \{c_1 < V(x) < c_2\}$

Note No guarantee on uniqueness of periodic orbit (only existence)

Ⓐ For example consider "harmonic oscillator"

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1$$



Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ and $M = \{C_1 \leq V(x) \leq C_2\}$ $C_2 > C_1 > 0$

Then M contains no eq. pt.

$$\dot{V} = \dot{x}_1 x_1 + \dot{x}_2 x_2 = x_2 x_1 - x_1 x_2 = 0$$

so start in M stay in M \Rightarrow Periodic orbit in M (actually a continuum)

(B) $\dot{x}_1 = -x_1 + x_2$

$$\dot{x}_2 = -x_1 - x_2$$

$$V = x_1^2 + x_2^2 \Rightarrow \dot{V} = x_1(-x_1 + x_2) + x_2(-x_1 - x_2) = -(x_1^2 + x_2^2) = -V$$

Solutions go to origin since $V \rightarrow 0$ as $t \rightarrow \infty$. Therefore, M is a trapping region but the eq. pt inside M is not unstable.

so Condition 1 is not satisfied, so P.B. is not applicable.

On the other hand:

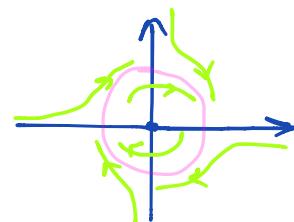
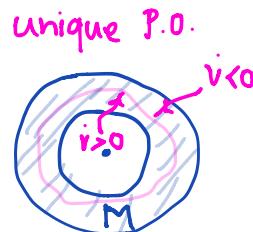
$$\text{div } f = -1 - 1 < 0 \xrightarrow[\text{criticism}]{\text{Bendixon}} \text{No P.O.}$$



(C) $\dot{x}_1 = x_2 + x_1(1 - x_1^2 - x_2^2)$

$$\dot{x}_2 = -x_1 + x_2(1 - x_1^2 - x_2^2)$$

$$\text{Let } V = x_1^2 + x_2^2 \text{ & } C_1 < 1 < C_2$$



& $M = \{x \in \mathbb{R}^2 \mid C_1 \leq V(x) \leq C_2\}$: annular region

$x = (0,0)$ is a unique unstable focus. But M does not contain o .

$$\dot{V} = x_1 x_2 + x_1^2(1 - x_1^2 - x_2^2) - x_1 x_2 + x_2^2(1 - x_1^2 - x_2^2) = V(1 - V)$$

$$\begin{aligned} \dot{V} > 0 & \text{ for } V < 1 \\ \dot{V} < 0 & \text{ for } V > 1 \end{aligned} \quad \left. \right\} \rightarrow M \text{ is positively invariant.}$$

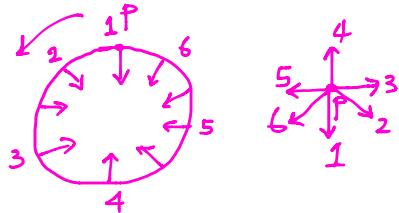
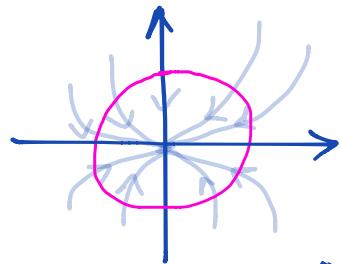
P.B. $\rightarrow \exists$ a p.o. inside M . Let $C_1 \& C_2 \rightarrow 1$ to see the orbit is unit circle.

Index Theory

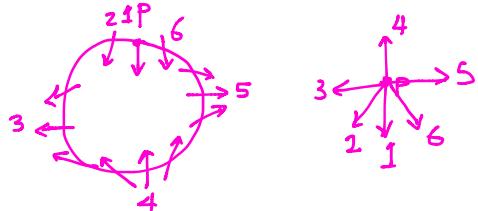
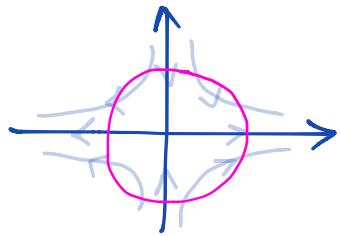
Def. Index k of a closed curve C , not passing through any eq. pt. is an integer defined as follows.

Let $f_{|C}$ be v.f. at $p \in C$. Let p traverse C in ccw direction back to initial position. The corresponding vector field $f(x)$ rotates an angle $2\pi k$, $k \in \mathbb{N}$.

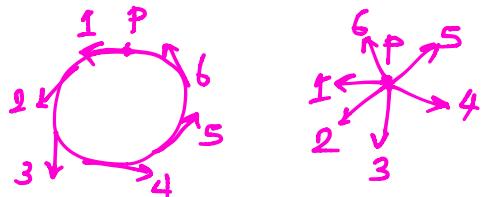
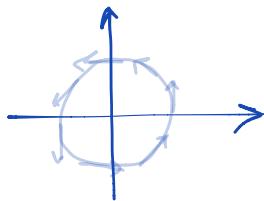
Example.



\circ : stable node $\rightarrow k = +1$



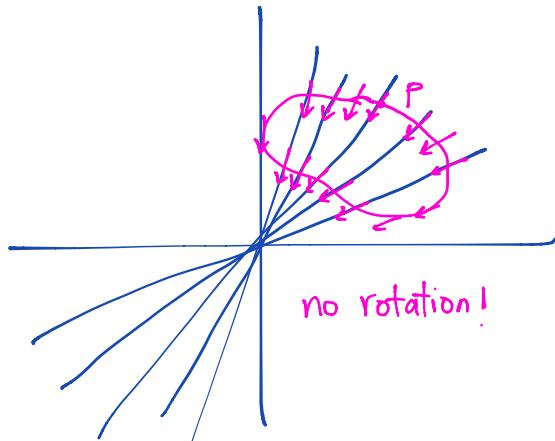
\circ : saddle $\rightarrow k = -1$



Periodic orbit $\rightarrow k = +1$

Lemma

- node, focus, center $\rightarrow k=+1$
- hyperbolic saddle $\rightarrow k=-1$
- closed orbits $\rightarrow k=+1$
- closed curve not encircling any eq. pt. $\rightarrow k=0$
- index of closed curve = \sum indices of all eq. pt. within it.



Corollary Inside any periodic orbit γ there must be at least one eq. pt. Suppose the eq. pts inside γ are hyperbolic. Then if N is the # of nodes & foci and S is the # of saddles, we have $N-S=1$.

- This is useful tool for ruling out periodic orbits in certain regions of the plane.

Example. $\dot{x}_1 = -x_1 + x_1 x_2$
 $\dot{x}_2 = x_1 + x_2 - 2x_1 x_2$ Has 2 eq. pts $(0,0)$ & $(1,1)$

1) eq. pt $(0,0)$ $\frac{\partial F}{\partial X} \Big|_{(0,0)} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \quad \lambda_{1,2} = \pm 1 \quad \text{saddle}$

2) eq. pt $(1,1)$ $\frac{\partial F}{\partial X} \Big|_{(1,1)} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad \lambda_{\pm} = \frac{-1 \pm i\sqrt{3}}{2} \quad \text{stable focus}$

If there is a periodic orbit, it must encircle stable focus $(1,1)$ & not the saddle.

Bifurcation (sec 2.7)

What happens when we perturb parameters in a system enough s.t. we drive equilibria to point of being structurally unstable?

Example.

Saddle-node bifurcation

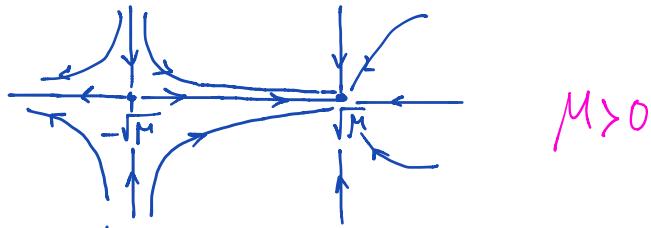
$$\begin{cases} \dot{x}_1 = \mu - x_1^2 \\ \dot{x}_2 = -x_2 \end{cases} \quad \mu \text{ a parameter}$$

Let $\mu > 0$. System has eq. pts $(\sqrt{\mu}, 0)$ & $(-\sqrt{\mu}, 0)$

Linearization: $\frac{\partial f}{\partial x} \Big|_{(\sqrt{\mu}, 0)} = \begin{pmatrix} -2\sqrt{\mu} & 0 \\ 0 & -1 \end{pmatrix} \rightarrow (\sqrt{\mu}, 0): \text{stable node}$

$$\frac{\partial f}{\partial x} \Big|_{(-\sqrt{\mu}, 0)} = \begin{pmatrix} 2\sqrt{\mu} & 0 \\ 0 & -1 \end{pmatrix} \rightarrow (-\sqrt{\mu}, 0): \text{saddle}$$

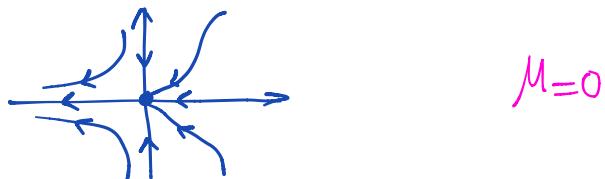
Phase portrait



Suppose we decrease μ all the way to $\mu=0$. Then at $\mu=0$,

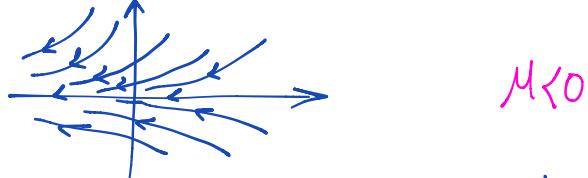
$$\begin{cases} \dot{x}_1 = -x_1^2 \\ \dot{x}_2 = -x_2 \end{cases} \quad \text{only one eq. pt } (0,0), \quad \frac{\partial f}{\partial x} \Big|_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

Phase portrait



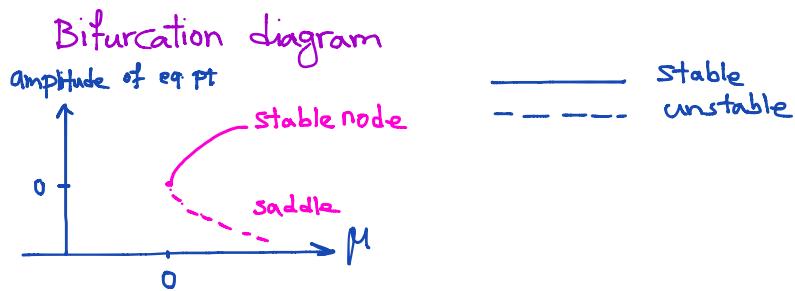
Suppose we continue to decrease μ so that $\mu < 0$. Then there are no eq. pt.

Phase portrait



Def. Bifurcation is change in the eq. pt or periodic orbits of a system or their stability properties as a (bifurcation) parameter is varied.

Def Value of bifurcation parameter at bifurcation is called bifurcation point. ($\mu=0$ above)



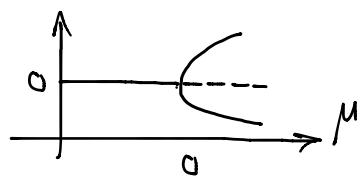
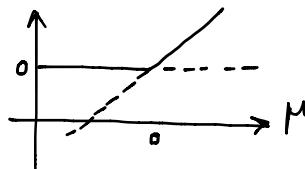
This bifurcation is called "saddle-node" bifurcation because it is the collision of saddle & node.

In this example bifurcation point corresponds to zero eigenvalue.

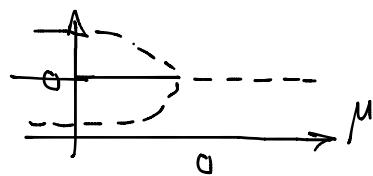
Very dramatic change in behavior (think about change in sign of very small magnitude μ) called hard (dangerous) bifurcation.

Next:

- Transcritical Bifurcation
- Pitchfork Bifurcation

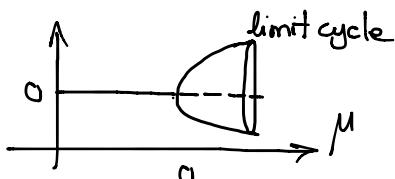


super critical

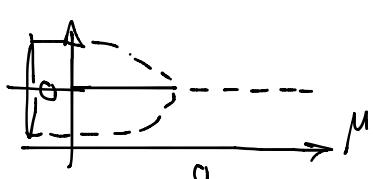


subcritical

- Hopf Bifurcation



super critical



subcritical

- 6 papers about Bifurcation posted in blackboard.